Fully nonlinear oblique derivative problems for singular degenerate parabolic equations on nonsmooth domain

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●要約

　本論文では平均曲率流方程式を含む退化放物型方程式の境界値問題を考察した。境界条件はオブリーグ境界値問題を含む非線形性の高い境界条件とする。考える領域の境界が滑らかな問題では、すでに粘性解の解の一意性および存在は証明されている。また領域の境界が滑らかでない場合のノイマン問題では、新たな粘性解の定義の下、解の一意性および存在の証明がなされている。本論文では、考える領域の境界が滑らかでない場合のノイマン境界値問題ですでに得られている解の一意性および存在定理の結果をオブリーク境界値問題を含む非線形境界値問題に拡張することに成功したことを示す。

●キーワード

degenerate parabolic
nonsmooth domain
comparison principle

1 Introduction

In this paper we are concerned with the following boundary value problem

(1.1) $$u_t + F(t, x, u, Du, D^2u) = 0 \qquad \text{in } Q = (0, T) \times \Omega,$$

(1.2) $$B(x, Du) = 0 \qquad \text{in } S = (0, T) \times \partial\Omega,$$

where Ω is a bounded domain in $\mathbf{R}^n$ and $T > 0$. Here $u_t = \partial u/\partial t$, and Du and D^2u denote, respectively, the gradient and Hessian of u. Let Ω be a bounded domain in $\mathbf{R}^n$ and $\Omega = \bigcap_{i \in I} \Omega_i$ where I is a finite index set and $\Omega_i' s$ are domains in $\mathbf{R}^n$ with relatively regular boundary such that $\partial\Omega_i \in C^1$. For $x \in \partial\Omega$ we denote by $I(x)$ the set of those indices i which satisfy $x \in \partial\Omega_i$. We deal with equations (1.1) in a class of singular degenerate parabolic equations which includes the mean curvature flow equation. In the case when F is continuous in its variables, there is already a comparison and existence result for viscosity solutions of second order degenerate parabolic PDE with boundary condition (1.2). We refer for this to [3]. In the case of singular PDE like the mean curvature flow equation and $\partial\Omega$ is smooth, Giga and Sato [4] have established comparison and existence results for viscosity solutions under the Neumann condition and the author [8], Ishii-Sato [5] and Barles [1] treated the case of fully nonlinear boundary condition including capillaly boundary condition.In [9] we have already proved comparison and existence theorems under the Neumann boundary condition in the case Ω is piecewise smooth. Our aim in this paper is to establish comparison and existence theorems concerning viscosity solutions of (1.1)-(1.2) when Ω is piecewise smooth.

This paper is organized as follows. At first we constluct the key test function and prove our comparison result and establish our existence result.

2 A comparison and existence theorem

We are concerned with the following boundary value problem

(2.1) $$u_t + F(t, x, u, Du, D^2u) = 0 \qquad \text{in } Q = (0, T) \times \Omega,$$

(2.2) $$B(x, Du) = 0 \qquad \text{in } S = (0, T) \times \partial\Omega,$$

where Ω is a bounded domain in $\mathbf{R}^n$ and $T > 0$. Here $u_t = \partial u/\partial t$, and Du and D^2u denote, respectively, the gradient and Hessian of u. Let Ω be a bounded domain in $\mathbf{R}^n$ and $\Omega = \bigcap_{i \in I} \Omega_i$ where I is a finite index set and $\Omega_i' s$ are domains in $\mathbf{R}^n$ with relatively regular boundary such that $\partial\Omega_i \in C^1$. For $x \in \partial\Omega$ we denote by $I(x)$ the set of those indices i which satisfy $x \in \partial\Omega_i$.

We start by listing our assumptions of F and B. Henceforth, for $p, q \in \mathbf{R}^n \setminus \{0\}$ we write $\overline{p} = \frac{p}{|p|}$ and $\rho(p, q) = \left[(|p| \wedge |q|)^{-1}|p - q|\right] \wedge 1$. Here and henceforth we use the notation: $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

(F1) $F \in C([0, T] \times \overline{\Omega} \times \mathbf{R} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n)$,

where $\mathcal{S}^n$ denotes the space of $n \times n$ real matrices equipped with the usual ordering.

(F2) There exists a constant $\gamma \in \mathbf{R}$ such that for each $(t, x, p, X) \in [0, T] \times \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n$ the function $u \mapsto F(t, x, u, p, X) - \gamma u$ is non-decreasing on $\mathbf{R}$.

(F3) For each $R > 0$ there exists a continuous function $\omega_R : [0,\infty) \to [0,\infty)$ satisfying $\omega_R(0) = 0$ such that if $X, Y \in \mathcal{S}^n$ and $\mu_1, \mu_2 \in [0,\infty)$ satisfy

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

then

$$\begin{aligned} &F(t,x,u,p,X) - F(t,y,u,q,-Y) \\ &\qquad \ge -\omega_R(\mu_1(|x-y|^2 + \rho(p,q)^2) + \mu_2 + |p-q| + |x-y|(|p| \vee |q| + 1)). \end{aligned}$$

for all $t \in [0,T],\ x,y \in \overline{\Omega},\ u \in \mathbf{R}$, with $|u| \le R$, and $p, q \in \mathbf{R}^n \setminus \{0\}$.

(B1) $B \in C(\mathbf{R}^n \times \mathbf{R}^n) \cap C^{1,1}(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$.

(B2) For each $x \in \mathbf{R}^n$ the function $p \mapsto B(x,p)$ is positively homogeneous of degree one in p, i.e., $B(x,\lambda p) = \lambda B(x,p)$ for all $\lambda \ge 0$ and $p \in \mathbf{R}^n \setminus \{0\}$.

(B3) There exists a positive constant θ such that $\langle \nu(z), D_p B(z,p) \rangle \ge \theta$ for all $z \in \partial\Omega$ and $p \in \mathbf{R}^n \setminus \{0\}$. Here $\nu(z)$ denotes the unit outer normal vector of Ω at $z \in \partial\Omega$.

(B4) For each $i \in I$ the boundary $\partial\Omega_i$ is of class C^1.

Theorem 2.1. (comparison principle) *Suppose that (F1)–(F3) and (B1)–(B4) hold. Let* $u \in \mathrm{USC}([0,T) \times \overline{\Omega})$ *and* $v \in \mathrm{LSC}([0,T) \times \overline{\Omega})$ *be, respectively, viscosity sub- and supersolutions of (1)–(2). If* $u(0,x) \le v(0,x)$ *for* $x \in \overline{\Omega}$, *then* $u \le v$ *on* $(0,T) \times \overline{\Omega}$.

Theorem 2.2. (existence)*Assume that (F1)–(F3) and (B1)–(B4) hold. Then for each* $u(0,x) = g \in C(\overline{\Omega})$ *there is a (unique) viscosity solution* $u \in C([0,T) \times \overline{\Omega})$ *of (1)–(2) .*

Let $Q_0 = (0,T) \times \overline{\Omega}$. A function $u : Q_0 \to \mathbf{R}$ is called a viscosity subsolution of (1.1)-(1.2) if it satisfies the following properties:

$$\begin{aligned} &\text{(i)} \quad u^* < +\infty \\ &\text{(ii)} \quad \tau + F_*(x,r,p,X) \le 0 \quad \text{for } x \in \Omega \quad (\tau,p,X) \in \rho^{2,+}_{Q_0} u^*(t,x) \\ &\quad \tau + F_*(x,r,p,X) \wedge \min\{B(x,p) : i \in I(x)\} \le 0 \\ &\quad \text{for} \quad x \in \partial\Omega \quad (\tau,p,X) \in \rho^{2,+}_{Q_0} u^*(t,x) \end{aligned}$$

Similarly a function $u : Q_0 \to \mathbf{R}$ is called a viscosity subsolution of (1.1)-(1.2) if it satisfies the following properties:

$$\begin{aligned} &\text{(i)} \quad u_* > -\infty \\ &\text{(ii)} \quad \tau + F^*(x,r,p,X) \ge 0 \quad \text{for} \quad x \in \Omega \quad (\tau,p,X) \in \rho^{2,-}_{Q_0} u_*(t,x) \\ &\quad \tau + F^*(x,r,p,X) \wedge \min\{B(x,p) : i \in I(x)\} \ge 0 \\ &\quad \text{for} \quad x \in \partial\Omega \quad (\tau,p,X) \in \rho^{2,-}_{Q_0} u_*(t,x) \end{aligned}$$

Here $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$ and $\rho^{2,+}_{Q_0} u^*(t,x)$ (resp.$\rho^{2,-}_{Q_0} u_*(t,x)$) denotes the parabolic super 2-jet in Q_0. (see [2]) Any function u

Remark 2.2. Assumptions (F1) and (F3) imply that

$$-\infty < F_*(t,x,u,0,0) = F^*(t,x,u,0,0) < \infty$$

Some typical examples of F satisfying (F1)-(F3) are picked up.

Let $A : \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \to M^{n\times m}$, where $M^{n\times m}$ denotes the space of real $n \times m$ matrices, be a continuous function which is homogeneous of degree zero, i.e.,

$$A(x, \lambda p) = A(x,p) \quad \text{for all } (x,p,\lambda) \in \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \times (0,\infty)$$

and which satisfies

$$\|A(x,p) - A(y,q)\| \le C_1(|x-y| + |p-q|)$$

for all $x, y \in \overline{\Omega}$ and $p, q \in S^{n-1}$, where $C_1 > 0$ is a constant and S^{n-1} denotes the unit sphere $\{\xi \in \mathbf{R}^n : |\xi| = 1\}$.

Let $b \in C(\overline{\Omega}, \mathbf{R}^n)$ satisfy

$$|b(x) - b(y)| \le C_2|x-y| \quad \text{for all } x, y \in \overline{\Omega}.$$

Furthermore let $c, f \in C(\overline{\Omega}, \mathbf{R})$ be given. Define the function $F \in C(\overline{\Omega} \times \mathbf{R} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^\setminus)$ by

$$F(x,u,p,X) = -\mathrm{tr}\,[A(x,p)A(x,p)^*X] + \langle b(x), p\rangle + c(x)u + f(x).$$

It is now easy to see that F satisfies condition (F3). Also, it is immediate to see that condition (F2) is satisfied with $\gamma \le \min_{\overline{\Omega}} c$. To check (F4), we note that for any $(t,x,r,p,X) \in [0,\,T] \times \overline{\Omega} \times \mathbf{R} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^\setminus$,

$$|F(t,x,r,p,X) - c(x)r - f(x)| \le nC_4\|X\| + C_5|p|,$$

where $C_5 = \max\{|b(x)| : x \in \overline{\Omega}\}$, and find that $F^*(t,x,r,0,0) = F_*(t,x,r,0,0) = c(x)r + f(x)$ for all $(t,x) \in [0,\,T] \times \overline{\Omega}$. Thus F satisfies (F1)–(F4).

If $A(x,p) = I - |p|^{-2}(p \otimes p)$, $b = 0$, and $c = f = 0$, then it is the case of the mean curvature flow equation and the above conditions on A, b, c, and f are valid.

Next we deal with the boundary condition. Consider the function B of the form

$$B(x,p) = \langle \mu(x), p\rangle - |C(x)p|,$$

where $\mu : \mathbf{R}^n \to \mathbf{R}^n$ is a $C^{1,1}$ vector field over $\mathbf{R}^n$ and $C : \mathbf{R}^n \to M^{n\times n}$ is a $C^{1,1}$ function satisfying $\det C(x) \neq 0$ in a neighborhood of $\partial\Omega$.

It is clear that (B2) is satisfied. We can modify the definition of B so that the resulting function $\widetilde{B}$ satisfies (B1) and $\widetilde{B}(x,\cdot) = B(x,\cdot)$ for all x in a neighborhood of $\partial\Omega$.

As before let $\nu(x)$ denote the unit outer normal of Ω at $x \in \partial\Omega$. By calculation, we have

$$D_pB(x,p) = \mu(x) - \frac{C(x)^*C(x)p}{|C(x)p|} \quad \text{if } p \neq 0,$$

and we see that (B3) is equivalent to the condition

$$\langle \mu(x), \nu(x)\rangle > \langle \xi, C(x)\nu(x)\rangle \quad \text{for all } (x,\xi) \in \partial\Omega \times S^{n-1}.$$

A particular case is when $\mu = \nu$ and $C(x) = a(x)I$ for some $a \in C^{1,1}(\mathbf{R}^n)$ such that $0 < a(x) < 1$ for $x \in \partial\Omega$, which corresponds to the capillary condition. In this case the boundary regularity of Ω should be of class $C^{2,1}$ in order that $\mu = \nu \in C^{1,1}(\mathbf{R}^n)$ is satisfied, which is one of requirements of Theorems 2.1 and 3.1. It is interesting to find that the results in [6,7] need the same $C^{2,1}$ regularity of the boundary.

Lemma 2.3.

Assume (B1)–(B4). *Then there are a positive constant σ and a function $v \in C^{1,1}(\overline{\Omega} \times \mathbf{R}^n)$ such that for all $(x, \xi) \in \overline{\Omega} \times \mathbf{R}^n$,*

$$v(x,\xi) \geq |\xi|^2,$$

$$v(x,\lambda\xi) = \lambda^2 v(x,\xi) \qquad \text{for all } \lambda \in [0,\infty),$$

$$B(x, D_\xi v(x,\xi)) \geq 0 \quad \text{if } x \in \partial\Omega \text{ and } \langle \nu(x), \xi\rangle \geq -\sigma|\xi|,$$

$$B(x, D_\xi v(x,\xi)) \leq 0 \quad \text{if } x \in \partial\Omega \text{ and } \langle \nu(x), \xi\rangle \leq \sigma|\xi|.$$

We can choose a function $\psi \in C^\infty(\overline{\Omega})$ having the properties:

$$\psi \geq 0 \quad \text{on} \quad \overline{\Omega} \quad \text{and} \quad \langle D_p B(x,p), D\psi(x)\rangle \geq 1 \quad \text{for} \quad (x,p) \in \partial\Omega \times (\mathbf{R}^n \setminus \{0\}).$$

Let v and σ be a function on $\overline{\Omega} \times \mathbf{R}^n$ and a positive constant, respectively, for which the conditions of Lemma 2.3 hold. We set $g = v^2$ and choose a constant $\tilde{C_1} > 0$ so that for a. e. $(x,\xi) \in \overline{\Omega} \times \mathbf{R}^n$,

$$g(x,\xi) \vee |D_x g(x,\xi)| \leq \tilde{C_1}|\xi|^4 \ \text{ and } \ |D_\xi g(x,\xi)| \vee \|D_x D_\xi g(x,\xi)\| \leq \tilde{C_1}|\xi|^3.$$

We fix a Lipschitz constant $M \geq 1$ of the function B on $\overline{\Omega}$, and set $\varphi(x) = M\tilde{C_1}\psi(x)$ for $x \in \overline{\Omega}$. Then we have

$$\langle D_p B(x,p), D\varphi(x)\rangle \geq M\tilde{C_1} \qquad \text{for} \qquad (x,p) \in \partial\Omega \times (\mathbf{R}^n \setminus \{0\}).$$

We define

$$u(x,y) = e^{\varphi(x)+\varphi(y)} g(x, x-y) \quad \text{ for } (x,y) \in \overline{\Omega} \times \overline{\Omega}.$$

Lemma 2.4.

Assume that (B1)–(B4) *hold. Then there are a function $w \in C^{1,1}(\overline{\Omega} \times \overline{\Omega})$ and positive constants C and δ such that for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,*

(i) $|x-y|^4 \leq w(x,y) \leq C|x-y|^4,$

$|D_x w(x,y)| \vee |D_y w(x,y)| \leq C|x-y|^3,$

(ii) $B(x, D_x w(x,y)) \geq 0 \quad$ *if* $x \in \partial\Omega,$

$B(y, -D_y w(x,y)) \leq 0 \quad$ *if* $y \in \partial\Omega,$

(iii) $|D_x w(x,y) + D_y w(x,y)| \leq C|x-y|^4,$

$\rho(D_x w(x,y), -D_y w(x,y)) \leq C|x-y| \quad$ *if* $0 < |x-y| \leq \delta,$

and for a. e. $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

(iv) $D^2 w(x,y) \leq C\Big\{|x-y|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + |x-y|^4 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\Big\}.$

In what follows we use the notation: for any $p, q \in \mathbf{R}^n$,

$$\rho^*(p,q) = \begin{cases} \rho(p,q) & \text{if } \ p, q \neq 0, \\ 1 & \text{if either } \ p = 0 \ \text{ or } \ q = 0. \end{cases}$$

Note that the function ρ^* is upper semi-continuous on $\mathbf{R}^n \times \mathbf{R}^n$.

Sketch proof of Theorem 2.1. We may assume that u and v are bounded on $[0,T) \times \overline{\Omega}$ and that the function $r \mapsto F(t,x,r,p,X)$ is non-decreasing in $\mathbf{R}$ for each $(t,x,p,X) \in [0,T] \times \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n$. (see [5])

By virtue of lemma 2.3, there are a function $w \in C^2(\overline{\Omega} \times \overline{\Omega})$ and a positive constant C such that for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

(2.3) $$|x-y|^4 \le w(x,y) \le C|x-y|^4,$$
$$|D_x w(x,y)| \vee |D_y w(x,y)| \le C|x-y|^3,$$

(2.4) $$\langle \gamma_i(x), D_x w(x,y) \rangle \ge 0 \qquad \text{for all } x \in \partial\Omega, \qquad i \in I(x)$$
$$\langle \gamma_i(y), -D_y w(x,y) \rangle \le 0 \qquad \text{for all } y \in \partial\Omega, \qquad i \in I(y)$$

(2.5) $$|D_x w(x,y) + D_y w(x,y)| \le C|x-y|^4,$$
$$\rho^*(D_x w(x,y), -D_y w(x,y)) \le C|x-y|,$$

and for a. e. $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

(2.6) $$D^2 w(x,y) \le C\Big\{ |x-y|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + |x-y|^4 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \Big\}.$$

We argue by contradiction. So we suppose that

(2.7) $$m_0 := \sup\{u(t,x) - v(t,x) \ : \ (t,x) \in [0,T) \times \overline{\Omega}\} > 0.$$

For $\alpha > 0, \ \varepsilon > 0, \ \delta > 0$ we define

$$\Psi(t,x,y) = \frac{\varepsilon}{T-t} + \alpha w(x,y) + \delta(\varphi(x) + \varphi(y)),$$
$$\Phi(t,x,y) = u(t,x) - v(t,y) - \Psi(t,x,y)$$

for $(t,x,y) \in [0,T) \times \overline{\Omega} \times \overline{\Omega}$. Here the function $\varphi \in C^2(\overline{\Omega})$ satisfies

$$\varphi > 0 \quad \text{on} \quad \overline{\Omega} \quad \text{and} \quad \langle D\varphi(x), \gamma_i(x) \rangle \ge 1 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad i \in I(x)$$

Actually we can construct the above function φ. (see [3]) From (2.7) we infer that for sufficiently small $\varepsilon > 0$ and $\delta > 0$, the function Φ attains a maximum greater that $m_0/2$. Fix such δ and ε, and choose a maximum point $(\hat{t}, \hat{x}, \hat{y})$ of Φ. Note that Φ and $(\hat{t}, \hat{x}, \hat{y})$ depend on $\alpha, \ \varepsilon, \ \delta$.

It is now well-known (see, e.g., [2]) that

(2.8) $$\lim_{\varepsilon \searrow 0} \lim_{\alpha \to \infty} \lim_{\delta \searrow 0} \Phi(\hat{t}, \hat{x}, \hat{y}) = m_0,$$

(2.9) $$\lim_{\alpha \to \infty} \sup\{\alpha w(\hat{x}, \hat{y}) \ : \ 0 < \delta < 1, \ 0 < \varepsilon < 1\} = 0. \qquad \square$$

Sketch proof of Theorem 2.2.

We use the Perron method (see [2]) to show the existence of a continuous viscosity solution of (2.1)–(2.2). For this, the first step is to build sub- and supersolutions of (2.1)–(2.2) satisfying initial data.

Fix any $0<\varepsilon<1$ and choose a constant $\alpha(\varepsilon)>0$ so that

$$|g(x)-g(y)|\le \varepsilon+\alpha(\varepsilon)|x-y|^4 \quad \text{for all } x,y\in\overline{\Omega}.$$

Let β be a positive function of $\varepsilon\in(0,\,1)$ to be fixed later, and we define functions $V^{\pm}$ on $[0,T)\times\overline{\Omega}$ parametrized by $\varepsilon\in(0,\,1)$, $y\in\overline{\Omega}$, respectively, by

$$\begin{aligned} V^{+}(t,x;\varepsilon,y) &= g(y)+\varepsilon+\alpha(\varepsilon)w(x,y)+\beta(\varepsilon)t,\\ V^{-}(t,x;\varepsilon,y) &= g(y)-\varepsilon-\alpha(\varepsilon)w(y,x)-\beta(\varepsilon)t. \end{aligned}$$

We can see that for any $y\in\overline{\Omega}$,

$$V^{-}(t,x;\varepsilon,y)\le g(x)\le V^{+}(t,x;\varepsilon,y) \quad \text{for all } x\in\overline{\Omega}.$$

We intend to select the function β so that V^+ and V^- are viscosity super- and subsolutions of (2.1)–(2.2), respectively. To do this, fix $(t,x)\in(0,T)\times\overline{\Omega}$ and let $(a,p,X)\in\rho^{2,-}_{Q_0}V^{+}(t,x)$. where $a\in\mathbf{R},p\in\mathbf{R^n},X\in\mathbf{S^n}$. (See [4] for the definition of parabolic semi-jets)

We first consider the case when $x\in\partial\Omega$. We then observe for a closely related observation that $p=\alpha(\varepsilon)D_xw(x,y)+r\nu(x)$ for some $r\ge0$. Using (B3) we see that $B(x,p)\ge0$. This shows that

$$B(x,p)\vee\big(a+F^{*}(t,x,V^{+}(t,x;\varepsilon,y),p,X)\big)\ge 0.$$

We next consider the case when $x\in\Omega$.

We can see that there exists constant $\tilde{C}>0$ such that

$$|p|\le\tilde{C},\qquad X\le\alpha(\varepsilon)\tilde{C}I,\qquad V^{+}(t,x;\varepsilon,y)\ge g(y)\ge-\tilde{C}.$$

Using (F2), with $\gamma=0$, the degenerate ellipticity of F^*, we find that

$$\begin{aligned} F^{*}(t,x,V^{+}(t,x;\varepsilon,y),p,X) &\ge F^{*}(t,x,-\tilde{C},p,\alpha(\varepsilon)\tilde{C}I)\\ &\ge F(t,x,-\tilde{C},e,0)-\omega_{\tilde{C}}((\alpha(\varepsilon)+1)\tilde{C}+1). \end{aligned}$$

We fix

$$\beta(\varepsilon)=\max_{(t,x)\in\overline{Q}}|F(t,x,\tilde{C},e,0)|\vee|F(t,x,-\tilde{C},e,0|+\omega_{\tilde{C}}((\alpha(\varepsilon)+1)\tilde{C}+1),$$

and observe that $a=\beta(\varepsilon)$ and hence

$$\begin{aligned} B(x,p)\ &\vee\ \big(a+F^{*}(t,x,V^{+}(t,x;\varepsilon,y),p,X)\big)\\ &\ge\ a+F^{*}(t,x,V^{+}(t,x;\varepsilon,y),p,X)\ge0. \end{aligned}$$

Consequently, with the above choice of function β, for each $(\varepsilon,y)\in(0,\,1)\times\overline{\Omega}$ the function $V^{+}(\cdot;\varepsilon,y)$ is a viscosity supersolution of (2.1)–(2.2). Similar considerations to the above show that, with the above choice of β, for each $(\varepsilon,y)\in(0,\,1)\times\overline{\Omega}$ the function $V^{-}(\cdot;\varepsilon,y)$ is a viscosity subsolution of (2.1)–(2.2).

Next, we define functions $f^{\pm}$ on $[0,T)\times\overline{\Omega}$ by

$$\begin{aligned} f^{+}(t,x) &= \inf\{V^{+}(t,x;\varepsilon,y)\,:\,0<\varepsilon<1,\ y\in\overline{\Omega}\},\\ f^{-}(t,x) &= \sup\{V^{-}(t,x;\varepsilon,y)\,:\,0<\varepsilon<1,\ y\in\overline{\Omega}\}. \end{aligned}$$

Then we easily deduce that $f^{\pm}$ are continuous on $[0,T)\times\overline{\Omega}$, $f^{-}(t,x)\le g(x)\le f^{+}(t,x)$ for all $(t,x)\in[0,T)\times\overline{\Omega}$ and $f^{\pm}(0,x)=g(x)$ for all $x\in\overline{\Omega}$ and that f^{+} and f^{-} are viscosity super- and subsolutions of (2.1)–(2.2), respectively.

Now we conclude by the Perron method together with Theorem 2.1 that if we define the function u on $[0,T)\times\overline{\Omega}$ by

$$u(t,x)=\sup\{v(t,x)\,:\,v\in S^{-}\},$$

where S^{-} denotes the set of functions v on $[0,T)\times\overline{\Omega}$ such that v is a viscosity subsolution of (2.1)–(2.2) and such that $f^{-}\le v\le f^{+}$ on $[0,T)\times\overline{\Omega}$, then u is a continuous function on $\overline{\Omega}\times[0,T)$, which is a consequence of Theorem 2.1, and a viscosity solution of (2.1)–(2.2). Noting that u satisfies $u(x,0)=g$, we conclude the proof. □

● References

[1] G. Barles, Nonlinear Neumann boundary conditions for quasilinear degenerate elliptic equations and applications. J. Differential Equations 154 (1999), no. 1, 191-224.

[2] M. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1-67.

[3] P.Dupuis and H.Ishii,On oblique derivative problems for fully nonlinear second-order elliptic PDE's on domains with corners. Hokkaido Mathematical Journal 20 (1991), 135-164

[4] Y. Giga and M.-H. Sato, Neumann problem for singular degenerate parabolic equations. Differential Integral Equations 6 (1993), no. 6,1217-1230.

[5] H. Ishii and M.-H. Sato, Nonlinear oblique derivative problems for singular degenerate parabolic equations on a general domain. Nonlinear Analysis 57 (2004)1077-1098

[6] H.Ishii and P.L.Lions, Viscosity solutions of fully nonlinear second -order elliptic partial differential equations. Journal of Differential Equations 83 (1990), 26-78

[7] P.L.Lions and A.S.Sznitman, Stochastics diferential equations with reflecting boundary conditions. Communicatins on Pure and Applied Mathematics (1984), 511-537

[8] M.-H. Sato, Capillary problem for singular degenerate parabolic equations on a half space. Differential Integral Equations 9 (1996), no. 6, 1213-1224.

[9] F.DA Lio and M.-H.Sato, Neumann problem for singular degenerate parabolic equations on nonsmooth domains 京都数理研究所講究録 vol.1323, pp174-183(2003)

●英文タイトル

Fully nonlinear oblique derivative problems for singular degenerate parabolic equations on nonsmooth domain

●要約

In this paper I consider the fully nonlinear oblique boundary problems for singular degenerate parabolic equations on nonsmooth domains. I define the weak solution for the fully nonlinear oblique boundary problem on the nonsmooth domain using the classical viscosity solution. In this paper I can establish comparison princeple and existence theorem for the above problems.

● Key words

degenerate parabolic
nonsmooth domain
comparison principle