Numerical analysis for the boundary value problem of the first order ordinary deferential equation

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●要約

一階のハミルトンヤコビ方程式の多くは、粘性解の一意存在性が知られている。一階のハミルトンヤコビ方程式には、最適制御問題や微分ゲームなどの様々な重要な方程式がある。しかし粘性解の数値計算による近似解を求める手法は確立していない。粘性解は２階の退化する楕円型方程式や放物型方程式に対しても有効であるが、またそれらの方程式に対する数値計算法も確立していない現状がある。２階の退化する楕円型方程式や放物型方程式に対しての数値計算への応用を念頭に、一階常微分方程式の境界値問題に対して幾つかの数値計算の手法を提示する。

●キーワード

viscosity solution
ordinary differential equation
uniqueness existense

1 Introduction

In this paper we are concerned with the following boundary value problem

(1.1) $$|\frac{du}{dx}| = 1 \qquad \text{on} \qquad -1 < x < 1,$$

(1.2) $$u(-1) = u(1) = 0.$$

Here $\frac{du}{dx}$ denotes the derivative of x. It is known that the equation (1.1)-(1.2) does not have classical solutions. The traditional mehod was to solve the following equation

(1.3) $$|\frac{du}{dx}| - 1 = \varepsilon \frac{d^2 u}{d^2 x} \qquad \text{on} \qquad -1 < x < 1,$$

(1.4) $$u(-1) = u(1) = 0,$$

To solve the euations (1.3)-(1.4) instesad of (1.1)-(1.2) was called vanishing viscosity method. We can find a classical solution u_ε of (1.3)-(1.4)

(1.5) $$u_\varepsilon(x) = \begin{cases} 1 - x + \varepsilon(e^{-\frac{1}{\varepsilon}} - e^{-\frac{x}{\varepsilon}}), & 0 \leq x \leq 1 \\ 1 + x + \varepsilon(e^{-\frac{1}{\varepsilon}} - e^{\frac{x}{\varepsilon}}), & -1 \leq x \leq 0. \end{cases}$$

This paper is organized as follows. At first we give the definition of viscosity solution for (1.1)-(1.2). We also prove the uniqueness and existence of the viscotisy solution for (1.1)-(1.2). The algorithm for deriving the viscosity solution is proposed in sec.3.

2 uniqueness and existence results

We define the viscosity subsolution and supersolution of (1.1)-(1.2). In this paper we give a strong sense definition of viscosity solution. In general we can relax the denitions of the viscosity solution (see[2],[4]).

Denition 2.1.

For all $-1 < x < 1$ we define the upper (resp.lower) derivative of u$\in C^{\circ}(-1,1)$

$$D^+ u(\hat{x}) = \{p \in \mathbf{R} | u(x) \leq u(\hat{x}) + p(x - \hat{x}) + o(|x - \hat{x}) \quad as \quad x \to \hat{x}\},$$

$$D^- u(\hat{x}) = \{p \in \mathbf{R} | u(x) \geq u(\hat{x}) + p(x - \hat{x}) + o(|x - \hat{x}) \quad as \quad x \to \hat{x}\}.$$

Definition 2.2. A continuous function $u : [-1,1] \to \mathbf{R}$ is called a viscosity subsolution of (1.1)-(1.2) if it satisfies the following properties:

(i) $u(-1) = u(1) = 0$

(ii) $|p| - 1 \leq 0$ for $p \in D^{+}u(x)$, $x \in (-1,1)$

Similarly a continuos function $u : [-1,1] \to \mathbf{R}$ is called a viscosity supersolution of (1.1)-(1.2) if it satisfies the following properties:

(i) $u(-1) = u(1) = 0$

(ii) $|p| - 1 \geq 0$ for $p \in D^{-}u(x)$, $x \in (-1,1)$

If u is both viscosity subsolution and a viscosity supersolution, then u is called a viscosity solution.

Remark 2.3.

For u_{ε} in (1.5) we can easily see that $|u_{\varepsilon} - (1-|x|)| \leq M\varepsilon$ where M is constant independent of ε. The clasical solution u_{ε} of (1.3)-(1.4) uniformly converges to $u_0(x) = 1-|x|$.

Theorem 2.4. (uniqueness and existence) The function $u_0(x) = 1-|x|$ is a unique viscosity solution of (1.1)-(1.2).

Sketch proof of Theorem 2.4. By the definition 2.1 we can calculate

(2.1) $$D^{+}u_0(x) = D^{-}u_0(x) = 1 \quad \text{on} \quad -1 < x < 0,$$

(2.2) $$D^{+}u_0(x) = D^{-}u_0(x) = -1 \quad \text{on} \quad 0 < x < 1,$$

(2.3) $$D^{+}u_0(0) = \{p \in \mathbf{R} \mid |p| \leq 1\},$$

(2.4) $$D^{-}u_0(0) = \emptyset.$$

From (2.1)-(2.4) we can easily see u_0 is a viscosity solution of (1.1)-(1.2). Next we put

(2.5) $$\tilde{u}(x) = \begin{cases} 1 - |x| & (x \neq 0), \\ C & (x = 0). \end{cases}$$

which $C\ (\neq 1)$ is constant.

We see the function $\tilde{u}(x)$ is not a viscosity solution of (1.1)-(1.2). Considering the results (2.1) and (2.2) u_0 is a unique viscosity solution of (1.1)-(1.2).

Remark 2.5.

In the case that the equation $H(Du) = 0$ is convex in Du, the viscosity solution is unique (see [5]). Here Du denotes the gradient of u.

3 Proposing the algorithm

In this section, we consider the viscosity solution algorithm. The approach for solving the problem is different from the convergence of the approximation by the vanishing viscosity method. We derive the viscosity solution by constraining ill-posed acts of equational solution. In the constraining, two verications narrow down the candidate of the solution; aspect of test-functions successively on every independent variable and satisfying the boundary values.

3.1 Numerical analysis of the differential equation

The numerical analysis is approximate means allowed for a margin of error because digital computer can not render infinitely-dividing. In the numerical analysis of the differential equation, the original equation $u'(x)$ is digitized by $u'[x]$ showing point sequence. The approximate solution is derived by sequential computing to population density of $u'[x]$. In consequence, the method needs boundary value and initial entry. As a result, the approximate solution shows point sequence of $u[x]$.

3.2 Derivative of discrete function

Our strategy of discrete derivative is to do subtraction between neighborhood point and remarkable point. This follows the denition of the continuous function. In any function $u[x]$, derivative $u'[x]$ at any $x = x_n$ ($n =$ 0,1,2,...) is a gradient in a minute interval $\Delta x = x_{n+1} - x_n$ as follows;

(3.1)
$$u'[x_n] = \frac{u[x_{n+1}] - u[x_n]}{\Delta x}.$$

However, $u'[x_n]$ by (3.1) makes shift length $\Delta x/2$ in x than continuous derivative $u'(x)$. To prevent the shift length problem, we redefine the framework of the discrete derivative $D_s u[x_n]$ as follows;

(3.2)
$$D_s u[x_n] = \frac{u[x_{n+1}] - u[x_{n-1}]}{x_{n+1} - x_{n-1}}.$$

In this paper, we call $D_s u[x_n]$ the“ slope ”. From here, the value of $u'[x_n]$ is described as the slope and $D_g u[x_n]$ denotes as the gradient value.

Although the slope is rough rate-of-variability than the gradient, its precision betters with $x \to 0$. Here, it is to be noted that the slope is denable at a indifferentiable point in continuous functions. In Figure 1, we show difference between“ slope ”and“ gradient ”.

3.3 Constraining to act of derivative

A part and parcel of the viscosity solution is maximum principle (or minimum principle). For deriving derivative to indifferentiable function, viscosity solution make the transition to the test function by the maximum principle.

On the basis of comprehension so far, from digitized equation, there are constraints to aspect of ill-posed solutions.

- The two boundaries $(x_s, u[x_s])$ and $(x_l, u[x_l])$ are evident. Here, $x_s < x_l$.
- There are point(s) with different gradient on either side. The point(s) $(x_0, u[x_0])$ are indifferentiable in $u(x)$.
- In the open interval $[x_s, x_l]$, the $u[x_0]$ is maximum of the $u[x]$.

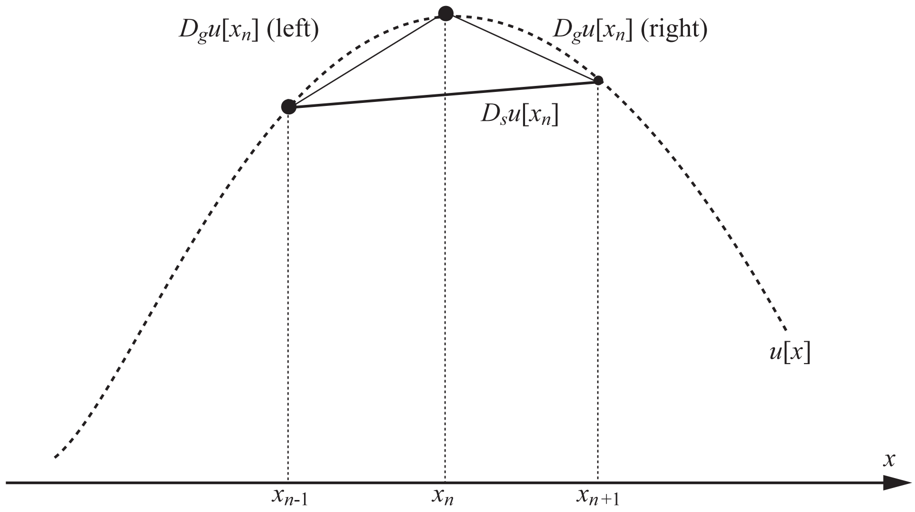

Figure 1: The “slope” and the “gradient”.

When the initial condition of differential equation is given and the equation's (x, $u[x]$) points are successively computed, the maximum of $u[x]$ is unknown. If the $u[x]$ is maximum at $x = x_0$ and $u(x)$ is in differentiable at (x_0, $u(x_0)$), let us consider relation between the $u[x]$ and following 2 curves as the test functions.. One is $g^+[x]$, this is always $g^+[x] \geq u[x]$. Another is $g^-[x]$, this is always $g^-[x] \leq u[x]$. Each of $g^\pm$ is a set with function, either $g^\pm[x_0]$ $= u[x_0]$ and another is approximate about $u[x]$. Here, $g^+[x]$ is supposition of the subordinate solutions and $g^-[x]$ is supposition of the dominated solutions.

We assume that acts of $u[x]$ is constrained by fitting these curves. The N-dimension curves contain simple parabola ($N = 2$), that are enough to constrain the $u[x]$ without a hitch. If the one of parabola is $C_{1g}[x]$, another is $-g[x]+C_2$. Here, $g[x]$ is turned on 2 end points of $u[x]$' domain and (x_0, $u[x_0]$). 2 constants are $C_1 \leq 1$ and $C_2 = 2$ ($u[x_0] - u[x_s]$) in general. Figure 2 shows the relation of $u[x]$, $g[x]$, one of g^+ and one of g^-.

Our concrete strategy is as follows.

First, we aim at (x_n, $u[x_n]$). If (x_n, $u[x_n]$) indifferentiable in $u(x)$, next point (x_{n+1}, $u[x_{n+1}]$) is assumable and $g^\pm$ $[x]$ are found in (x_n, $u[x_n]$). In verifying a point (x_n,$u[x_n]$), some sets of such (x,$u[x]$) might be produced including the possibility that g is considered at the following points.

Next, We apply rejecting-conditions to each (x,$u[x]$) candidate set. When $g^+[x]$ or $g^-[x]$ does not satisfy the denition of the viscosity solution, such (x,$u[x]$) set is not a candidate. As further rejecting-conditions, we check a possibility that $u[x]$ arrives at (x_l, $u[x_l]$). In the verifying a (x,$u[x]$) set successively, if the set is not able to arrive at (x_l, $u[x_l]$), such (x, $u[x]$) set is not a candidate, too.

3.4 Algorithm

For a differential equation $u'[x] = 0$, there have to be a distance of the Δx and 2 boundaries. One bound-ary (x_s,$u[x_s]$) is also the initial condition. Anothor boundary (x_l, $u[x_l]$) is the end point of successive computing. Our algorithm is as follows.

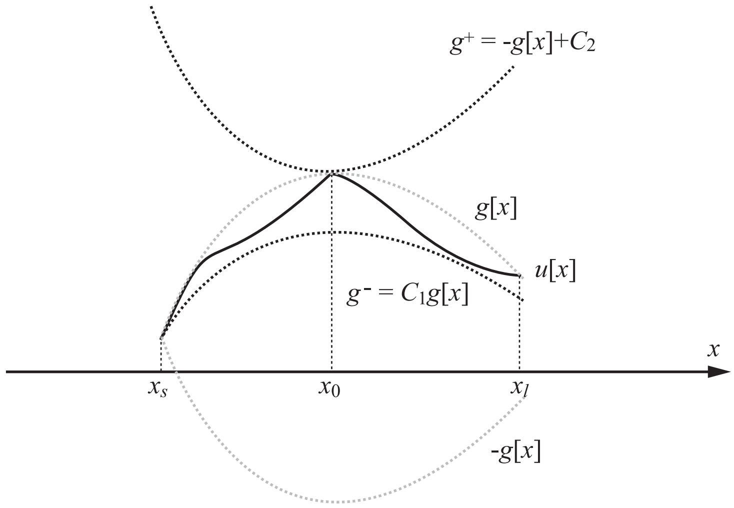

Figure 2: The aspect of each test function.

Step 1 For $x_l = x_s + \Delta x$ and $x_2 = x_l + \Delta x$, plot x_1 and x_2 under the condition of the $D_s u[x_1]$. A point sequence Ω = $\{(x_s, u[x_s]), (x_1, u[x_1]), (x_2, u[x_2])\}$ is a candidate of the $u[x]$. However, there might be some Ω s.

Step 2 Plot x_2 differ than step1 under the condition of the $D_g u[x_1]$ because $u(x)$ might be indifferentiable at x_1. Such point sequence(s) Ω_u is a candidate of the $u[x]$, too.

Step 3 For each Ω_u, make a parabola $g[x]$ by using 3 points $(x_s, u[x_s])$, $(x_2, u[x_2])$ and the indifferentiable point of $u(x)$. If $C_g[x]$ or $-g[x] + C$ is not the test function under the denition of the viscosity solution, reject the Ω_u from candidate.

Step 4 Check whether each Ω and Ω_u arrives at $(x_l, u[x_l])$ by acs of $u[x]$. If an Ω (Ω_u) is not able to arrive at $(x_l, u[x_l])$, reject the Ω (Ω_u) from candidate.

Step 5 For each remaining Ω and Ω_u, attend to $(x_2, u[x_2])$ and plot next $(x_3, u[x_3])$ as with Step 1 and Step 2. Here, the Ω s and the Ω_u s are updating.

Step 6 Renew candidate of the $u[x]$ as with Step 3 and Step 4.

Step 7 Repeat Step 5— Step 6 from $(x_3, u[x_3])$ to $(x_l, u[x_l])$. Rate a remainded Ω (or Ω_u) as the viscosity solution of the equation.

Here, we solve a differential equation, using (1.1) as an example. The bounderies are $(-1,0)$ and $(1,0)$. The initial condition is $(-1,0)$. For simplicity, we divide the domain into four Δxs (i.e. the $\Delta x = 0.5$). Each sequence of point $\Omega_n = \{(x_s, u[x_s]), (x_l, u[x_l]), (x_2, u[x_2]), ..., (x_l, u[x_l])\}$ (n = 0, 1, 2...) is a derived candidate.

First, in Step 1— Step3, there are four combinations of x_1 and x_2 as follows (see Figure 3);

$$\Omega_0 = \{(-1,0), (-0.5, 0.5), (0,1)\},$$
$$\Omega_1 = \{(-1,0), (-0.5, -0.5), (0, -1)\},$$
$$\Omega_2 = \{(-1,0), (-0.5, 0.5), (0,0)\},$$
$$\Omega_3 = \{(-1,0), (-0.5, -0.5), (0,0)\}.$$

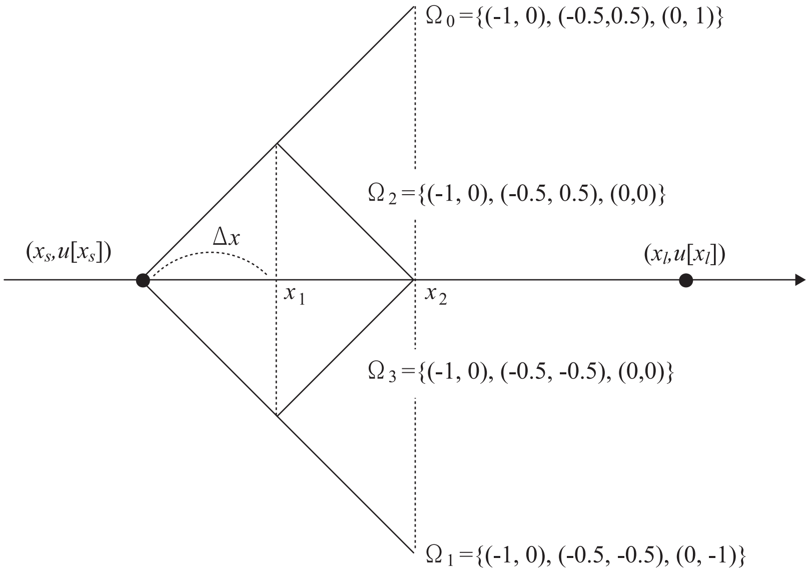

Figure 3: The initial condition of the equation (1.1).

Where, $\Omega_3 \in \Omega_u$ is not a candidate of solution because its test functions does not satisfy the viscosity solution in $[x_s, x_2]$. $\Omega_2 \in \Omega_u$ satisfies the requirement. Ω_0 and $\Omega_1 \in \Omega$ are not indifferentiable yet.

Next, in Step 4, Ω_0, Ω_1 and Ω_2 are able to arrive at $(x_l, u[x_l])$ via $(x_3, u[x_3]) = (0.5, 0.5)$ or $(0.5, -0.5)$.

Next, in Step 5, by $(x_3, u[x_3])$, each Ω_n is updated as follows (see Figure 4);

$$\Omega_0 = \{(-1, 0), (-0.5, 0.5), (0, 1), (0.5, 1.5)\},$$

$$\Omega_1 = \{(-1, 0), (-0.5, -0.5), (0, -1), (0.5, -1.5)\},$$

$$\Omega_2 = \{(-1, 0), (-0.5, 0.5), (0, 0), (0.5, 0.5)\},$$

$$\Omega_4 = \{(-1, 0), (-0.5, 0.5), (0, 1), (0.5, 0.5)\},$$

$$\Omega_5 = \{(-1, 0), (-0.5, -0.5), (0, -1), (0.5, -0.5)\},$$

$$\Omega_6 = \{(-1, 0), (-0.5, 0.5), (0, 0), (0.5, -0.5)\},$$

Where, Ω_2, Ω_5 (at x_2) and Ω_6 (at x_3) are not candidate as with Ω_3. Moreover, Ω_0 and Ω_1 are not candidate because these are not able to arrive the $(x_l, u[x_l])$ by next act of $u[x]$.

At last, only $\Omega_4 = \{(-1, 0), (-0.5, 0.5), (0, 1), (0.5, 0.5), (1, 0)\}$ remains as the candidate of the viscosity solution.

3.5 Considering

To be approximate well in $u(x)$, the domain has to consist of many Δxs for $u[x]$. In our proposed algorithm, a distance of the Δx exerts an influence to arrive $(x_l, u[x_l])$ successfully. In sec. 3.4, $u[x]$ arrived $(x_l, u[x_l])$ by dividing a domain in four Δxs. If the domain consists of five Δxs, $u[x]$ is not able to arrive $(x_l, u[x_l])$.

The algorithm does not expect appropriate Δx's distance for striking $(x_l, u[x_l])$. Therefore, $u[x_l]$ should have error margin. Additionally, verifying $u[x]$ in inverse direction from $(x_l, u[x_l])$ is desirable for satisfying the boundaries condition. In the“ bilateral verifying ”, estimating the crossing point's value is indispensable because each direction verifying has accumulative error (see Figure 5).

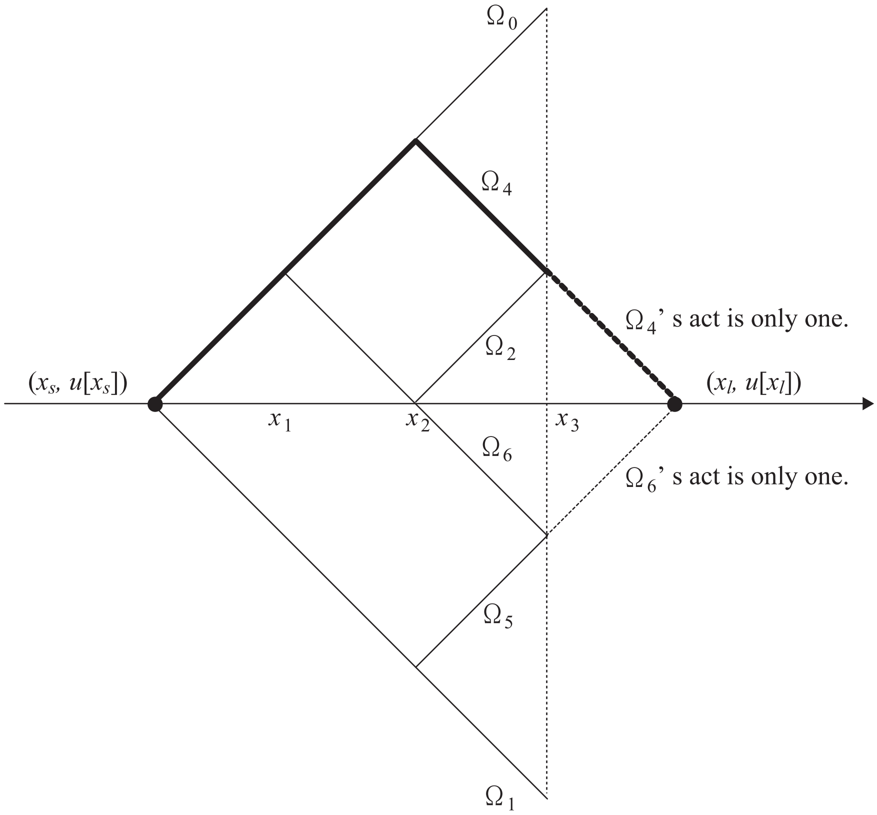

Figure 4: The candidates of the $u[x]$ in x_3.

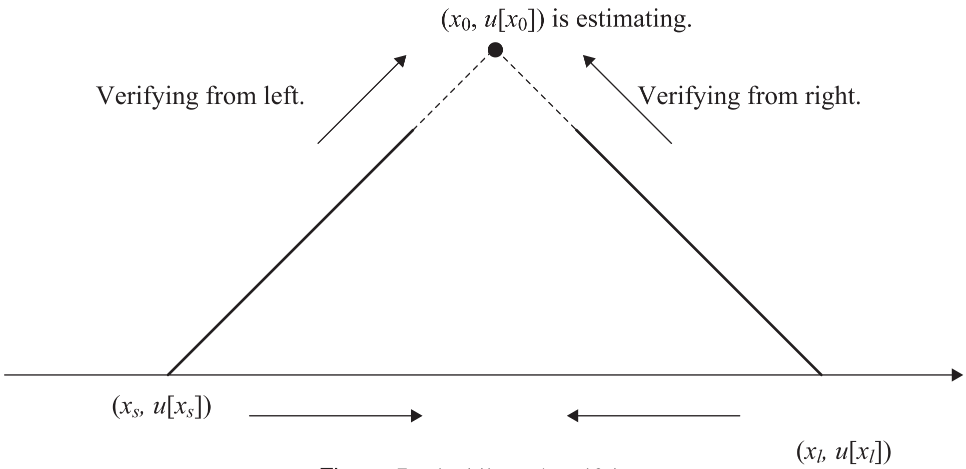

Figure 5: The bilateral verifying.

4 Conclusion

In this paper, for the boundary value problem of the first order ordinary deferential equation, we described about the viscosity solution that is a solving method derives the unique solution. And we proposed the algorithm for deriving the viscosity solution. The algorithm solved a typical first order ordinary deferential equation successfully. In the future problem, we will develop the bilateral verifying, and extend the algorithm for second order degenerate elliptic or parabolic equations.

●References

[1] M.G.Crandall and P.L.Lions, Viscosity solutions of Hamilton-Jacobiequations Trans. Amer. Math Soc., 277 (1983) 1-42

[2] M. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1-67.

[3] 儀我美一・陳蘊剛， 動く曲面を追いかけて，日本評論社

[4] 石井仁司，非線形偏微分方程式の粘性解について，数学，46（1994）144-157

[5] P.L.Lions, Generalized solutions of Hamilton-Jacobi equations Researc Notes in Math. vol.69, Pitman, Boston, 1982

●要約

The visconsity solution was derived for a unique solution of the Hamilton-Jacobi equation by Crandall and Lions. However, the numerical analysis method for the visconsity solution is unestablished.

We consider the boundary problem for the first order ordinary differential equation. In this paper we prove the existence of a unique viscosity solution.

Our algorithm solve a typical first order ordinary deferential equation successfully.